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LETTER TO THE EDITOR

A physical interpretation of the quantum group $\mathcal{U}_q(\mathrm{SU}(2))$

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Abstract. The quantum statistics of a system of free spins in the presence of a constant magnetic field H is interpreted in terms of the representation theory of the simplest quantum group $\mathcal{U}_q(\mathrm{SU}(2))$ with $q = \mathrm{e}^{(\mu_0\mu_B/k_BT)}$ being the deformation parameter which has a definite physical meaning. The classical limit $q \to 1$ corresponds to weak magnetic field regimen $H \to 0$.

Recently, since the discovery of quantum groups (Hopf algebras [1]) by Drinfeld [2] as the natural algebraic setting for the inverse scattering problem, these algebraic structures have been shown to be deeply rooted in many problems of physical and mathematical interest, such as rational conformal field theories (RCFT) [3–6], exactly solvable statistical models [7], inverse scattering theory applied to integrable models in quantum field theories [8], non-commutative geometry [9], knot theory in three dimensions etc. In all these disparate areas of mathematical physics the Yang–Baxter equation plays an essential role.

In this letter, a simple interpretation of the quantum group $\mathcal{U}_q(\mathrm{SU}(2))$ is addressed. Namely, the effect of a constant magnetic field on a system of non-interacting spins localized at certain sites of space can be visualized as a q-deformation of the classical algebra SU(2) of angular momentum theory, with the deformation parameter q having a definite physical meaning. In this way, the quantum statistics of this system of spins can be interpreted in terms of the representation theory of the quantum group $\mathcal{U}_q(\mathrm{SU}(2))$. In this respect, this application of $\mathcal{U}_q(\mathrm{SU}(2))$ should be viewed as a 'toy-model' or as a 'q-riosity'.

The quantum Lie algebra $\mathcal{U}_q(\mathrm{SU}(2))$ is a deformation of the universal enveloping algebra of SU(2) which is endowed with a Hopf algebra structure. The quantum algebra $\mathcal{U}_q(\mathrm{SU}(2))$ can be characterized by giving its three generators J_+, J_-, J_z together with the following defining relations based on the Chevalley basis of SU(2):

$$[J_z, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}} \equiv [2J_z]_q \tag{2}$$

where q is the parameter of the deformation of the classical algebra SU(2). It is a real number or it has unit modulus in order to be compatible the adjoint operation

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$(J_+^\dagger = J_-, J_-^\dagger = J_+)$ with the algebra structure (1), (2). As usual, it is convenient to introduce q-numbers denoted by $[x]_q$:

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \xrightarrow{q\to 1} x. \tag{3}$$

The algebra SU(2) is recovered from (1), (2) in the limit $q \to 1$.

The Hopf algebra structure of $\mathcal{U}_q(\mathrm{SU}(2))$ is given by the existence of a comultiplication Δ, a coinverse γ and a co-unit ε defined by

$$\Delta(J_z) = J_z \otimes 1 + 1 \otimes J_z \tag{4}$$

$$\Delta(J_\pm) = J_\pm \otimes q^{H_z} + q^{-J_z} \otimes J_\pm \tag{5}$$

$$\gamma(J_z) = -J_z \qquad \gamma(J_\pm) = -q^\pm J_\pm \tag{6}$$

$$\varepsilon(1) = 1 \qquad \varepsilon(J_\pm) = \varepsilon(J_z) = 0. \tag{7}$$

The operation Δ is an algebra homomorphism. In this letter we are interested in the interpretation of Δ as the analogue of angular momentum composition. When $q = 1$ it turns out

$$\Delta(\boldsymbol{J}) = \boldsymbol{J} \otimes 1 + 1 \otimes \boldsymbol{J}. \tag{8}$$

This comultiplication (8) also provides a Hopf algebra structure to $\mathcal{U}(\mathrm{SU}(2))$ but while this is co-commutative, (4), (5) is not. In this sense $\mathcal{U}_q(\mathrm{SU}(2))$ is a non-trivial Hopf algebra.

Because of the resemblance between the algebraic structures of SU(2) and $\mathcal{U}_q(\mathrm{SU}(2))$, the representation theory of the quantum group is quite similar to the classical theory. Several authors [10, 12, 13] have proved that there exist irreps of $\mathcal{U}_q(\mathrm{SU}(2))$ labelled with $j = 0, \frac{1}{2}, 1, \ldots$ acting on a Hilbert space V^j with basis vectors

$$|jm\rangle_q \qquad -j \leqslant m \leqslant j \tag{9}$$

as follows

$$J_z|jm\rangle_q = m|jm\rangle_q \tag{10}$$

$$J_\pm|jm\rangle_q = \sqrt{[j \mp m]_q [j \pm m + 1]_q}\, |jm \pm 1\rangle_q. \tag{11}$$

We readily see from (11) that the usual numbers have turned into q-numbers. The irrep V^j has dimension $2j+1$. However, it is convenient to introduce the concept of q-dimension of the irrep V^j as follows

$$\dim_q V^j \equiv [2j+1]_q. \tag{12}$$

This q-dimension plays a very important role in the representation theory of quantum groups when q is a root of unity [5, 6]. Moreover, the multiplication of q-dimensions satisfies a Clebsch–Gordan rule [5, 15]:

$$[2j_1+1]_q \times [2j_2+1]_q = \sum_{j=|j_1-j_2|}^{j_1+j_2} [2j+1]_q. \tag{13}$$

The example comes from the quantum statistical study of paramagnetism. A simplified study [16] of this phenomenon can be modelled by means of a system of spins j localized on certain sites in the space and without interactions among themselves. When we say 'system of spins' we mean 'particles' whose unique defining property is its spin, and they have neither other internal structure nor they are moving. Under these circumstances, we can consider that the unique interaction among the spins is the coupling of their angular momenta $\boldsymbol{J}$. Each particle of spin-j in the system is associated with an irrep V^j of SU(2). Then the coupling of two particles j_1, j_2 with angular momenta $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ respectively, in order to produce a state j of angular momentum $\boldsymbol{J}=\boldsymbol{J}_1+\boldsymbol{J}_2$ *is given by the Clebsch–Gordan decomposition of the tensor product of* the irreps V^{j_1} and V^{j_2}:

$$V^{j_1}\otimes V^{j_2}=\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V^j \tag{14}$$

that we can graphically represent as the interaction diagram of figure 1.

Now let us introduce in the system of spins a constant magnetic field $\boldsymbol{H}$ in the z axis direction. We know that the magnetic moments $\boldsymbol{\mu}$ of each particle tend to be oriented in the direction of the applied magnetic field. But what we want to know is *how the coupling rule of angular momenta has been modified by the presence of the* magnetic field $\boldsymbol{H}$.

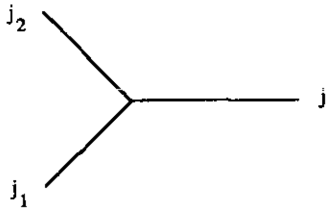

Figure 1. Interacting diagram for angular momentum coupling (Clebsch–Gordan decomposition).

In order to interpret the new situation it is convenient to remember the computation of the partition function $\mathscr{Z}_m$ of the whole system. This quantity encodes all the physical information of the system. First let us begin with the partition function z_m of one-particle states in the canonical ensemble:

$$z_m=\sum_{m=-j}^{+j} e^{-\beta\varepsilon_m} \tag{15}$$

where $\beta=1/k_B T$ and ε_m is the potential energy of the interaction of a magnetic dipole with moment $\boldsymbol{\mu}$ with the magnetic field $\boldsymbol{H}$:

$$\varepsilon_m=-\mu_0\boldsymbol{\mu}\cdot\boldsymbol{H}. \tag{16}$$

The magnetic moment is proportional to the angular momenta of the particles†

$$\boldsymbol{\mu}=g\frac{e}{2m_e}\boldsymbol{J} \tag{17}$$

where g is the Landé factor that equals 2 in this case. Then

$$\varepsilon_m=-2\mu_0\mu_B mH \tag{18}$$

† We will assume that the particles of spin-j we are considering have mass m_e and charge e.

with $\mu_B = e\hbar/2m_e$ the Bohr magneton. Substituting (18) in (15) we obtain a geometric series whose result can be written in the following way:

$$z_m = \frac{q^{2j+1} - q^{-(2j+1)}}{q - q^{-1}} \tag{19}$$

with

$$q \equiv (\mu_0 \mu_B / e^{k_B T}) \tag{20}$$

a real number. The partition function z_m can be interpreted in terms of the q-dimension of the irrep V^j of $\mathcal{U}_q(SU(2))$ introduced in (12):

$$z_m = [2j+1]_q = \dim_q V^j. \tag{21}$$

Alternatively, we can arrive at this result within the theory of quantum groups computing the q-dimension of the irrep V^j as a trace of the operator q^{2J_z} over the space of states $|jm\rangle_q$†

$$\dim_q V^j = \mathrm{tr}\, q^{2J_z} \tag{22}$$

then, $\dim_q V^j = \Sigma_{m=-j}^{+j} \langle jm|q^{2J_z}|jm\rangle_q = [2j+1]_q, (\hbar = 1)$ which is the same result obtained for the partition function z_m if we again identify q as in (20).

Then it is reasonable to interpret that the effect of introducing the magnetic field H in the system of spins has been to deformate the algebra of SU(2) to $\mathcal{U}_q(SU(2))$ with a deformation parameter given by (20). In this way the particles of spin-j transforms now with the representation V^j of $\mathcal{U}_q(SU(2))$. We observe from (20) that in the limit when the magnetic field disappears ($H \to 0$) the parameter of the deformation tends to 1 ($q \to 1$). In this way the algebra and representations of SU(2) are recovered. Let us recall that the limit of weak magnetic fields‡ corresponds to the classical theory of paramagnetism (Curie regimen).

Let us go now to consider the total partition function $\mathcal{Z}_m$ of the system of spins. First let us compute it as usual in statistical physics. As the particles of spin-j do not interact with each other and are localized at definite sites of space, we can apply the Maxwell-Boltzmann statistics as a good approximation [16]. Then

$$\mathcal{Z}_m = z_m^N. \tag{23}$$

Again we can arrive at the same result within representation theory of $\mathcal{U}_q(SU(2))$ with the interpretation mentioned above. In this context, the system composed of several spins is represented by the tensor product of the corresponding irreps V^j associated with each particle of spin-j. Therefore, let us denote by $j_1, j_2, \ldots, j_N$ the spins of the N particles, all of them equal to j but we shall suppose they are arbitrary for the moment. We begin adding to the system of only one particle j_1, with partition function given by (21), another particle of spin j_2. This system is represented by $V^{j_1} \otimes V^{j_2}$. The total spin k_1 (see figure 2(a)) can take values from $k_1 = |j_1 - j_2|$ to $k_1 = j_1 + j_2$, as follows from the Clebsch-Gordan decomposition associated with (14). The partition function of the compound system is obtained adding all the contributions of the configurations V^{k_1} coming from the angular momenta decomposition:

$$\mathcal{Z}_m = \sum_{k_1=|j_1-j_2|}^{j_1+j_2} [2k_1+1]_q \qquad V^{j_1} \otimes V^{j_2} = \bigoplus_{k_1=|j_1-j_2|}^{j_1+j_2} V^{k_1}. \tag{24}$$

† In fact, the q-dimension of V^j is a particular case of a Markov trace defined over the Hecke algebras constructed from the quantum group [6].

‡ Or low temperature as well.

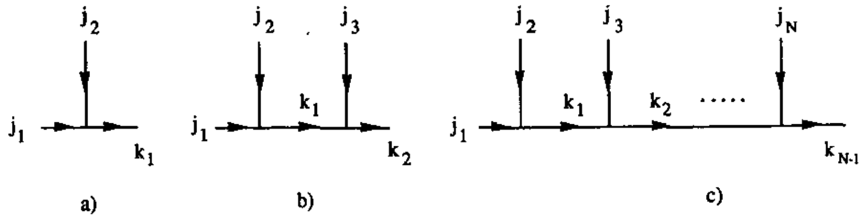

Figure 2. Interacting diagrams for coupling of 2, 3, ..., N angular momenta.

Using now the Clebsch–Gordan property of the q-dimensions (13) it turns out that the total partition function $\mathscr{Z}_m$ for the system of two spins is factorized:

$$\mathscr{Z}_m = [2j_1+1]_q[2j_2+1]_q \tag{25}$$

Now we add a third particle j_3 to the system that will be represented by $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ (see figure 2(*b*)). Again to compute $\mathscr{Z}_m$ we sum all the contributions coming from the partition functions associated to all the possible couplings of three spins:

$$\mathscr{Z}_m = \sum_{k_1=|j_1-j_2|}^{j_1+j_2} \sum_{k_2=|k_1-j_3|}^{k_1+j_3} [2k_2+1]_q = [2j_1+1]_q[2j_2+1]_q[2j_3+1]_q \tag{26}$$

where we have used again the property (13) to obtain the sum over all the possible contributions. The total partition function $\mathscr{Z}_m$ of a system of N spins $V^{j_1}\otimes V^{j_2}\otimes\ldots\otimes V^{j_N}$ is now readily generalized and it results:

$$\begin{aligned}\mathscr{Z}_m &= \sum_{k_1=|j_1-j_2|}^{j_1+j_2} \sum_{k_2=|k_1-j_3|}^{k_1+j_3} \cdots \sum_{k_{N-1}=|k_{N-2}-j_N|}^{k_{N-2}+j_N} [2k_{N-1}+1]_q \\ &= [2j_1+1]_q[2j_2+1]_q\ldots[2j_N+1]_q.\end{aligned} \tag{27}$$

Recalling that all the spins have the same value j, equation (27) reduces to (23) as desired.

The total partition function can be also expressed by means of a trace. Let $V^j_{\otimes N}$ denote the tensor product of N copies of the irreps V^j representing the total system. Then, it can be readily seen that

$$\mathscr{Z}_m = \mathrm{tr}(q^{2J_z}\otimes\ldots\otimes q^{2J_z})_{V^j_{\otimes N}} \tag{28}$$

has the same value as the one given by (27).

Let us notice that when angular momenta are coupled in the presence of a magnetic field, the couplings are carried out by 2×2 spins in such a way that to arrive the final configuration several options can be chosen. For instance, let us consider the coupling of three particles $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ as in figure 2(*b*). There are two possibilities to cluster the three spins. One is related to the configuration $(V^{j_2}\otimes V^{j_2})\otimes V^{j_3}$ (see figure 3(*a*)) and the other one to $V^{j_1}\otimes(V^{j_2}\otimes V^{j_3})$ (figure 3(*b*)). Both descriptions of the coupling of three particles are physically equivalent. In fact, if we denote by $e^{j_{12}j}_m(j_1j_2|j_3)$ an orthonormal basis in $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ associated with the scheme of figure 3(*a*) and $e^{j_{23}j}_m(j_1|j_2j_3)$ the one associated with the scheme of figure 3(*b*), we know from ordinary angular momentum theory that both bases are equivalent to each other and they are related by the q-analogue of 6j-symbols $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q$:

$$e^{j_{12}j}_m(j_1j_2|j_3) = \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}_q e^{j_{23}j}_m(j_1|j_2j_3). \tag{29}$$

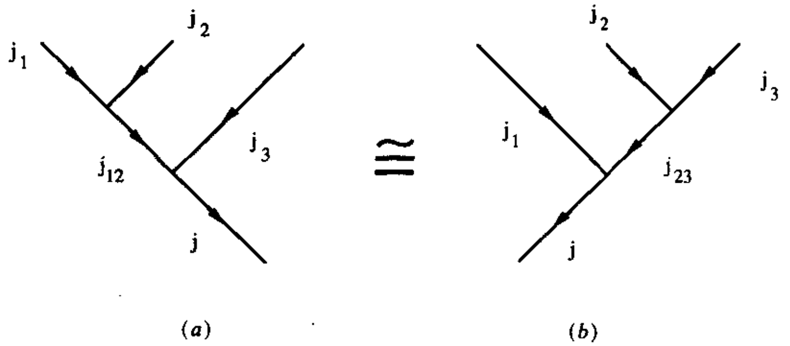

Figure 3. (*a*) $(V^{j_1}\otimes V^{j_2})\otimes V^{j_3}$, (*b*) $V^{j_1}\otimes(V^{j_2}\otimes V^{j_3})$.

As far as the total partition function is concerned, both coupling configurations of j_1, j_2, j_3 must be physically equivalent. In fact, their partition functions $\mathscr{Z}_m^{(12)}$ and $\mathscr{Z}_m^{(23)}$ are identical:

$$\mathscr{Z}_m^{(12)} = \sum_{j_{12}=|j_1-j_2|}^{j_1+j_2} \sum_{j=|j_{12}-j_3|}^{j_{12}+j_3} [2j+1]_q = \sum_{j_{23}=|j_2-j_3|}^{j_2+j_3} \sum_{j=|j_{23}-j_1|}^{j_{23}+j_1} [2j+1]_q = \mathscr{Z}_m^{(23)} \tag{30}$$

as can be readily proved using relation (13) in order to verify that both equal $\mathscr{Z}_m = [2j_1+1]_q[2j_2+1]_q[2j_3+1]_q$ as in (25).

From the quantum group $\mathscr{U}_q(\mathrm{SU}(2))$ point of view what we are checking is the co-associativity property of the comultiplication Δ (4), (5) (or coupling rule of spins in the presence of a magnetic field in our case):

$$(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta. \tag{31}$$

This relation is defined over the triple tensor product $\mathscr{A}\otimes\mathscr{A}\otimes\mathscr{A}$ with $\mathscr{A} = \mathscr{U}_q(\mathrm{SU}(2))$. In this way we see the quantum group structure through the representation theory interpretation of the quantum statistics for the system of spins in the magnetic field. Furthermore, it is well known that classical $6j$-symbols satisfy certain relations among themselves. One of them is the Biedenharn–Elliot identity [12] whose q-analogue reads as follows:

$$\sum_d \begin{Bmatrix} j_2 & a & d \\ j_1 & c & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & d & e \\ j_1 & f & c \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ a & e & d \end{Bmatrix}_q = \begin{Bmatrix} j_{23} & a & e \\ j_1 & f & b \end{Bmatrix}_q \begin{Bmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{Bmatrix}_q. \tag{32}$$

This is known as the pentagonal equation in rational conformal field theories (RCFT) [3] where the $q6j$-symbols are interpreted as duality matrices for conformal blocks [5]. In the representation theory of quantum groups this is a translation of the defining relation of a quantum group [8]:

$$T_1 T_2 \mathscr{R} = \mathscr{R} T_2 T_1 \tag{33}$$

where $\mathscr{R}$ is the universal $\mathscr{R}$-matrix and $T_1 = T\otimes 1$, $T_2 = 1\otimes T$ with T denoting the generators of the quantum group. The universal $\mathscr{R}$-matrix is defined as the following endomorphism on the tensor product $\mathscr{A}\otimes\mathscr{A}$:

$$\mathscr{R}: \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}\otimes\mathscr{A}$$

$$\sigma\cdot\Delta(a) = \mathscr{R}\Delta(a)\mathscr{R}^{-1} \qquad \forall a\in\mathscr{A} \tag{34}$$

where $\sigma: \mathscr{A}\otimes\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ is the permutation map $\sigma(a_1\otimes a_2)=a_2\otimes a_1$ and it turns out that $\Delta'=\sigma\cdot\Delta$ is another comultiplication in the same algebra with antipode $\gamma'=\gamma^{-1}$. When the $\mathscr{R}$-matrix exists the Hopf algebra $\mathscr{A}$ becomes a quasitriangular Yang–Baxter algebra. This is because the compatibility condition for (34) to hold is the quantum Yang–Baxter equation (QYBE) [8]:

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23}=\mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12}. \tag{35}$$

The existence of the $\mathscr{R}$-matrix has a simple meaning in the scheme of the interacting spins with the magnetic field. It is the statement that the representations $V^{j_1}\otimes V^{j_2}$ and $V^{j_2}\otimes V^{j_1}$ are equivalent, with the equivalence given by the $\mathscr{R}$-matrix as in (34), that is, it is the physical requirement of interchangeability in the composition of physical systems.

It is remarkable to notice that the QYBE is reflected in the representation theory of $\mathscr{U}_q(\mathrm{SU}(2))$ by means of the so-called hexagonal equation in RCFT [3–5] which is the following q-analogue of a classical identity [18] satisfied by the $6j$-symbols [12]:

$$\begin{aligned}\sum_g (-1)^{a-b-g-f} q^{c_a-c_b-c_g-c_f} \begin{Bmatrix} h_2 & a & g\\ j_1 & c & b\end{Bmatrix}_q \begin{Bmatrix} j_3 & g & e\\ j_1 & d & c\end{Bmatrix}_q \begin{Bmatrix} j_3 & a & f\\ j_2 & e & g\end{Bmatrix}_q \\ =\sum_g (-1)^{d-c-g-e} q^{c_d-c_c-c_g-c_e} \begin{Bmatrix} j_3 & b & g\\ j_2 & d & c\end{Bmatrix}_q \begin{Bmatrix} j_3 & a & f\\ j_1 & g & b\end{Bmatrix}_q \begin{Bmatrix} j_2 & f & e\\ j_1 & d & g\end{Bmatrix}_q\end{aligned} \tag{36}$$

with $c_j=j(j+1)$, $c_a=a(a+1),\ldots$ the classical Casimir eigenvalues. There is a graphical representation of equation (36) which is the Reidermeister 3rd movement [12].

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